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The Asymptotic Solution of Linear Differential Equations of the Second Order for Large Values of a Parameter

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THE ASYMPTOTIC SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER FOR LARGE VALUES OF A PARAMETER

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CONTENTS

Asymptotic solutions of the differential equations

$$d^2w/dz^2 = \{uz^n + f(z)\}\, w \quad (n = 0, 1)$$

for large positive values of u, have the formal expansions

$$w = P(z)\left\{1 + \sum_{s=1}^{\infty} \frac{A_s(z)}{u^s}\right\} + \frac{P'(z)}{u} \sum_{s=0}^{\infty} \frac{B_s(z)}{u^s},$$

where P is an exponential or Airy function for $n = 0$ or 1 respectively. The coefficients $A_s(z)$ and $B_s(z)$ are given by recurrence relations. This paper proves that solutions of the differential equations exist whose asymptotic expansions in Poincaré's sense are given by these series, and that the expansions are uniformly valid with respect to the complex variable z. The method of proof differs from those of earlier writers and fewer restrictions are made.

1. Introduction and summary

The purpose of this investigation is to determine asymptotic expansions of solutions of the differential equation

$$\frac{d^2w}{dz^2} = \{up(z) + q(z)\}\, w \tag{1·1}$$

for large values of the parameter u. It is supposed that z lies in a simply-connected domain **D** and $p(z)$, $q(z)$ are analytic functions of the complex variable z independent of u. Attention will be confined to positive values of u; if u is a large complex parameter of the form $u = |u|\, e^{i\vartheta}$, where ϑ is a fixed real number, then the transformation $z' = e^{\frac{1}{2}i\vartheta} z$ converts (1·1) into the same form of equation but with u replaced by $|u|$.

The asymptotic character of the solutions of (1·1) as $u \to \infty$ can take many different forms depending on the number and nature of the *transition points* of (1·1) in **D**, such points being defined here† as ones at which $p(z)$ vanishes, or $p(z)$ or $q(z)$ has a singularity.

Many papers have been published on this subject. Nevertheless, the existing theory does not yet appear to be complete even in the cases when (1·1) has no transition points or only one transition point, a simple zero of $p(z)$. In the more recondite cases previous work has generally been concerned only with the determination of the asymptotic *form* of the solutions in the immediate neighbourhood of various kinds of transition points, and asymptotic *expansions* have not been investigated.

It may be thought that the form of equation (1·1) is unduly restrictive, particularly as other writers have considered more general forms, for example

$$\frac{d^2w}{dz^2} = \left\{up(z) + \sum_{s=0}^{\infty} \frac{q_s(z)}{u^s}\right\} w. \tag{1·2}$$

The principal reason for adopting the present form of equation is that many important second-order differential equations are already in the form (1·1), or may be readily transformed into it. Such equations include those of Bessel, Weber and Legendre, and new asymptotic expansions of standard solutions of these equations for large orders have been obtained, and it is hoped to give these in subsequent papers.

Three cases will be considered here, and they will be referred to as cases A, B and C. They occur respectively when equation (1·1) has, in **D**, (i) no transition points, (ii) one transition point, a simple zero of $p(z)$, (iii) one transition point, a double pole of $p(z)$. The expansions in the three cases are similar and it is convenient to treat them together. Similar expansions also exist when the only transition point is simultaneously a simple pole of $p(z)$ and a double pole of $q(z)$, but consideration of this important case is deferred.

The paper is arranged as follows. In § 2 it is shown that (1·1) may be transformed into a similar equation with $p(z)$ replaced by unity in cases A and C, and by z in case B. In § 3 series are constructed which formally satisfy these standard forms of equation. In § 4 some relevant properties of the exponential and Airy functions are recorded. In § 5 two existence theorems are stated which reveal the asymptotic nature of the series constructed in § 3. These theorems are the main results of the paper; their application is briefly considered in § 6. In § 7 the work of other writers is outlined. The remaining sections, §§ 8 to 11, are devoted to the proofs of the theorems stated in § 5.

2. Transformation to standard form

In (1·1) let us take a new independent variable ζ and a new dependent variable W, related by

$$W = \dot{z}^{-\frac{1}{2}} w, \tag{2·1}$$

dots denoting differentiations with respect to ζ. Then it is readily verified that W satisfies the equation

$$\frac{d^2W}{d\zeta^2} = \{u\dot{z}^2 p(z) + f(\zeta)\} W, \tag{2·2}$$

† Points at which $p(z)$ vanishes have been called 'turning points' by some writers (for example, Langer 1949) and transition points by others (Watson 1944, p. 248; and Cherry 1950). The wider definition of transition point used here is a useful generalization.

where
$$f(\zeta) \equiv \dot{z}^2 q(z) + \dot{z}^{\frac{1}{2}} \frac{d^2}{d\zeta^2}(\dot{z}^{-\frac{1}{2}}) = \dot{z}^2 q(z) + \frac{3\ddot{z}^2 - 2\dot{z}\dddot{z}}{4\dot{z}^2}. \tag{2·3}$$

The relation between z and ζ is now prescribed in the following ways.

Case A. Here $p(z)$ has no zeros in **D**. We take $\dot{z}^2 p(z) = 1$, so that
$$\zeta = \int \{p(z)\}^{\frac{1}{2}} dz. \tag{2·4}$$

Equation (2·2) accordingly becomes
$$\frac{d^2 W}{d\zeta^2} = \{u + f(\zeta)\} W. \tag{2·5}$$

The relation (2·4) maps the domain **D** on a certain domain **G** in the ζ-plane, and it is clear from (2·3) and (2·4) that $f(\zeta)$ is a regular (holomorphic) function of ζ in **G**.

Case B. Here $p(z)$ has a simple zero in **D** at $z = z_0$, say, and the transformation (2·4) is unsuitable because $dz/d\zeta$ becomes infinite there. Instead, we take $\dot{z}^2 p(z) = \zeta$ and
$$\zeta = \left[\frac{3}{2}\int_{z_0}^{z} \{p(z)\}^{\frac{1}{2}} dz\right]^{\frac{2}{3}}. \tag{2·6}$$

Near $z = z_0$ we have
$$\zeta = \text{constant} \times (z - z_0) + (z - z_0)^2 O(1). \tag{2·7}$$

Equation (2·2) becomes
$$\frac{d^2 W}{d\zeta^2} = \{u\zeta + f(\zeta)\} W, \tag{2·8}$$

where $f(\zeta)$ is given by (2·3) and is a regular function of ζ in the transformed domain.

Case C. Here $p(z)$ has a double pole in **D** at $z = z_0$, and we may allow $q(z)$ to have a single or double pole at the same point. The same transformation (2·4) is made as in case A above, so that the transformed equation is again given by (2·5). Near $z = z_0$ we have
$$p(z) = (z - z_0)^{-2} \{k^2 + (z - z_0) O(1)\}, \tag{2·9}$$

where k is a non-zero constant, and so
$$\zeta = k \ln (z - z_0) + (z - z_0) O(1). \tag{2·10}$$

Thus $z = z_0$ corresponds to $\zeta = -\infty \exp(i \arg k)$. It is seen from (2·3) that $f(\zeta)$ is again a regular function of ζ in the corresponding ζ-domain. The transition point is now at infinity, but a sufficient condition that the asymptotic expansions of solutions of (2·5) remain uniformly valid at infinity is (see § 5)
$$f(\zeta) = O(|\zeta|^{-2}), \quad \text{as} \quad |\zeta| \to \infty. \tag{2·11}$$

From (2·10) and (2·3) we obtain
$$z - z_0 \sim e^{\zeta/k}, \quad f(\zeta) = \frac{e^{2\zeta/k}}{k^2} q(e^{\zeta/k} + z_0) + \frac{1}{4k^2} + O(e^{\zeta/k}) = c + O(e^{\zeta/k}),$$

as $\zeta \to -\infty \exp(i \arg k)$, where
$$c = \frac{1}{k^2}[(z - z_0)^2 q(z)]_{z=z_0} + \frac{1}{4k^2}.$$

Thus equation (2·5) itself may violate the condition (2·11), but by making the linear change of variables
$$u_1 = u + c, \quad f_1(\zeta) = f(\zeta) - c,$$

we arrive at the equation $$\frac{d^2W}{d\zeta^2} = \{u_1 + f_1(\zeta)\}W,$$

which does satisfy the condition.

Thus in all three cases the original differential equation may be transformed into one of the *standard forms* (2·5) or (2·8) in which $f(\zeta)$ is a regular function. Accordingly, we shall confine our attention to these standard forms. It will be convenient to revert to the symbols w, z and $\mathbf{D}$ in place of W, ζ and $\mathbf{G}$ respectively.

The foregoing transformation for case B was first given by Langer (1931). That for case A is much older and was used by Liouville (1837).

3. Formal construction of series solutions

We consider now the standard form

$$\frac{d^2w}{dz^2} = \{ug(z) + f(z)\}w, \tag{3·1}$$

where $g(z) = 1$ for cases A and C, and z for case B.

Let $w = P(z)$ be any solution of the equation

$$\frac{d^2w}{dz^2} = ug(z)\,w. \tag{3·2}$$

(When $g(z) = 1$, P is an exponential function, and when $g(z) = z$ it is an Airy function.) Then as a possible solution of (3·1) we consider the series

$$w(z) = P(z)\left\{1 + \sum_{s=1}^{\infty}\frac{A_s(z)}{u^s}\right\} + \frac{P'(z)}{u}\sum_{s=0}^{\infty}\frac{B_s(z)}{u^s}, \tag{3·3}$$

where the coefficients $A_s(z)$ and $B_s(z)$ are independent of u. Term-by-term differentiation, using (3·2), yields

$$w'(z) = P(z)\sum_{s=0}^{\infty}\frac{C_s(z)}{u^s} + P'(z)\left\{1 + \sum_{s=1}^{\infty}\frac{D_s(z)}{u^s}\right\}, \tag{3·4}$$

and $$w''(z) = uP(z)\sum_{s=0}^{\infty}\frac{E_s(z)}{u^s} + P'(z)\sum_{s=0}^{\infty}\frac{F_s(z)}{u^s}, \tag{3·5}$$

where $$C_s = A_s' + gB_s, \quad D_s = A_s + B_{s-1}', \tag{3·6}$$

and $$\left.\begin{aligned} E_s &= C_{s-1}' + gD_s = A_{s-1}'' + gA_s + 2gB_{s-1}' + g'B_{s-1}, \\ F_s &= C_s + D_s' = 2A_s' + B_{s-1}'' + gB_s, \end{aligned}\right\} \tag{3·7}$$

primes denoting differentiations with respect to z.

Substituting (3·3) and (3·5) in (3·1), we see that the differential equation is formally satisfied if

$$E_{s+1} = gA_{s+1} + fA_s, \quad F_{s+1} = gB_{s+1} + fB_s. \tag{3·8}$$

Substituting E_{s+1} and F_{s+1} from (3·7), we obtain

$$A_s'' - fA_s + 2gB_s' + g'B_s = 0, \quad 2A_{s+1}' + B_s'' - fB_s = 0. \tag{3·9}$$

These equations may be integrated to give

$$B_s = \tfrac{1}{2}g^{-\frac{1}{2}}\int g^{-\frac{1}{2}}(fA_s - A_s'')\,dz, \quad A_{s+1} = -\tfrac{1}{2}B_s' + \tfrac{1}{2}\int fB_s\,dz. \tag{3·10}$$

Equations (3·10) may be regarded as recurrence relations for the coefficients $A_s(z)$ and $B_s(z)$. Taking $A_0 = 1$ we can determine all the higher coefficients, apart from the presence of certain arbitrary integration constants.

If the series (3·3) converged uniformly with respect to z then the process of term-by-term differentiation would be valid, and (3·3) would define a solution of (3·1). In general, however, the series diverges and it is natural to examine the manner and circumstances in which (3·3) represents a solution of (3·1). As a preliminary to this purpose some relevant facts about the Airy and exponential functions will now be stated.

4. Some properties of the Airy† and exponential functions

The Airy functions are the solutions of the differential equation

$$\frac{d^2w}{dz^2} = zw. \tag{4·1}$$

One standard solution is the well-known function

$$\mathrm{Ai}\,(z) = \frac{z^{\frac{1}{2}}}{\pi\sqrt{3}} K_{\frac{1}{3}}(\zeta), \tag{4·2}$$

where $\zeta = \frac{2}{3}z^{\frac{3}{2}}$. It is an integral function of z, and for large $|z|$ it takes the asymptotic values (Watson 1944, p. 202)

$$\mathrm{Ai}\,(z) = \frac{z^{-\frac{1}{4}}}{2\sqrt{\pi}} e^{-\zeta}\{1+O(|\zeta|^{-1})\}, \quad \mathrm{Ai}'\,(z) = -\frac{z^{\frac{1}{4}}}{2\sqrt{\pi}} e^{-\zeta}\{1+O(|\zeta|^{-1})\}, \tag{4·3}$$

when $|\arg z| < \pi$, and

$$\left.\begin{aligned} \mathrm{Ai}\,(-z) &= \frac{z^{-\frac{1}{4}}}{\sqrt{\pi}}\{\cos(\zeta-\tfrac{1}{4}\pi)+e^{|\mathscr{I}\zeta|}O(|\zeta|^{-1})\},\\ \mathrm{Ai}'\,(-z) &= \frac{z^{\frac{1}{4}}}{\sqrt{\pi}}\{\sin(\zeta-\tfrac{1}{4}\pi)+e^{|\mathscr{I}\zeta|}O(|\zeta|^{-1})\}, \end{aligned}\right\} \tag{4·4}$$

when $|\arg z| < \frac{2}{3}\pi$. The expressions (4·3) and (4·4) are equivalent in their common regions of validity. From them we may deduce the useful inequalities

$$\left.\begin{aligned} |\mathrm{Ai}\,(z)| &< A(1+|z|^{\frac{1}{4}})^{-1}\,|\exp(-\tfrac{2}{3}z^{\frac{3}{2}})|,\\ |\mathrm{Ai}'\,(z)| &< A(1+|z|^{\frac{1}{4}})\,|\exp(-\tfrac{2}{3}z^{\frac{3}{2}})|, \end{aligned}\right\} \tag{4·5}$$

valid when $|\arg z| \leqslant \pi$, where A is some constant.

The functions $\mathrm{Ai}\,(z\,e^{\pm\frac{2}{3}\pi i})$ also satisfy (4·1), and we shall find it convenient to use the notation

$$P_1(z) = \mathrm{Ai}\,(z), \quad P_2(z) = \mathrm{Ai}\,(z\,e^{\frac{2}{3}\pi i}), \quad P_3(z) = \mathrm{Ai}\,(z\,e^{-\frac{2}{3}\pi i}). \tag{4·6}$$

Any two of these functions comprise a linearly independent pair of solutions of (4·1), the Wronskians being

$$W(P_1,P_2) = \frac{e^{-\frac{1}{6}\pi i}}{2\pi}, \quad W(P_1,P_3) = \frac{e^{\frac{1}{6}\pi i}}{2\pi}, \quad W(P_2,P_3) = \frac{i}{2\pi}. \tag{4·7}$$

If the sectors $\quad |\arg z| \leqslant \frac{1}{3}\pi, \quad -\pi \leqslant \arg z \leqslant -\frac{1}{3}\pi, \quad \frac{1}{3}\pi \leqslant \arg z \leqslant \pi,$

are denoted by $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$ respectively (figure 1), then it is seen from (4·3) and (4·6) that $P_j(z)$ is exponentially small in $\mathbf{S}_j$ $(j = 1, 2, 3)$, when $|z|$ is large. Consequently in $\mathbf{S}_j$ any

† A fuller account of the properties of the Airy functions in the complex plane is given in the Appendix to the following paper.

independent pair of solutions of (4·1) must include $P_j(z)$ as a member if it is to be satisfactory from the numerical standpoint.†

Another standard solution of (4·1) is the function

$$\mathrm{Bi}\,(z) = \mathrm{e}^{\frac{1}{6}\pi \mathrm{i}}\, P_2(z) + \mathrm{e}^{-\frac{1}{6}\pi \mathrm{i}}\, P_3(z), \tag{4·8}$$

introduced in the *British Association Mathematical Tables* (1946). Although there is no region in which Bi (z) is exponentially small, it is important because it is real when z is real and is a numerically satisfactory† second solution to Ai (z) for both positive and negative z; its large negative zeros lie asymptotically half-way between those of Ai (z).

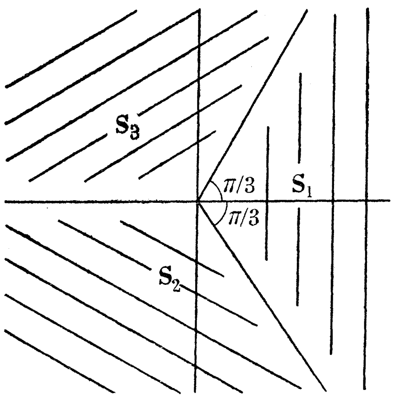

FIGURE 1. Regions in which $P_1(z)$, $P_2(z)$ and $P_3(z)$ are exponentially small.

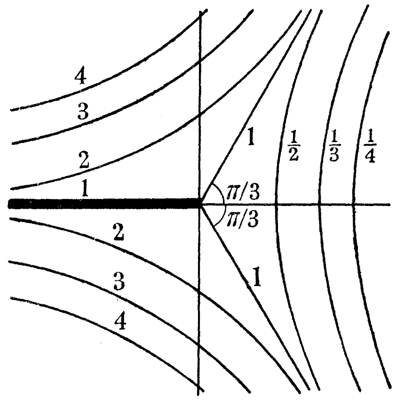

FIGURE 2. Level curves of $\exp(-\frac{2}{3}z^{\frac{3}{2}})$. Values of $|\exp(-\frac{2}{3}z^{\frac{3}{2}})|$ are indicated on the curves.

Level curves. The level curves of a function $F(z)$ are the contours along which

$$|F(z)| = \text{constant}.$$

In later sections we are concerned with the level curves of the functions $\exp z$ and $\exp(-\frac{2}{3}z^{\frac{3}{2}})$. Those of the former are straight lines parallel to the imaginary axis; those of $\exp(-\frac{2}{3}z^{\frac{3}{2}})$ have the polar equation

$$r^{\frac{3}{2}} \cos \tfrac{3}{2}\theta = \text{constant}, \tag{4·9}$$

where $z \equiv r\,\mathrm{e}^{\mathrm{i}\theta}$. Typical curves are illustrated in figure 2.

5. Asymptotic nature of the series; theorems A and B

The asymptotic nature of the series given in § 3 may be described by means of two existence theorems. In this and the following two sections the theorems are stated and their application described, the proofs being deferred until §§ 8 to 11.

In the following it is supposed that $f(z)$ is regular in a simply-connected open domain **D**, the boundaries of which consist of a finite number of straight lines; if **D** is unbounded it is supposed that‡

$$f(z) = O(|z|^{-2}), \tag{5·1}$$

as $|z| \to \infty$ in **D**, uniformly with respect to $\arg z$. We denote by **D**′ any simply-connected domain lying wholly in **D**, the boundaries of which consist of a finite number of straight lines not intersecting the boundaries of **D**.

† For a discussion of criteria for numerically satisfactory solutions of second-order differential equations see Miller (1950).

‡ There may of course be domains extending arbitrarily near infinity in which (5·1) is not satisfied.

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In the first theorem we suppose a_j $(j = 1, 2)$ to be an arbitrary fixed point of $\mathbf{D}'$; if $\mathbf{D}'$ is unbounded a_j can be the point at infinity on a straight line lying in $\mathbf{D}'$. In addition, we denote by $\mathbf{D}_j$ the domain comprising those points z of $\mathbf{D}'$ which can be joined to a_j by a contour which lies in $\mathbf{D}'$ and is *wholly to the left* if $j = 1$, or *wholly to the right* if $j = 2$, of the line through z parallel to the imaginary axis.

THEOREM A. *Let sequences of functions $A_s(z)$ and $B_s(z)$ be defined*† *by the relations* $A_0(z) = 1$,

$$A_{s+1}(z) = -\tfrac{1}{2}A_s'(z) + \tfrac{1}{2}\int f(z)\, A_s(z)\, \mathrm{d}z + \text{constant}, \tag{5·2}$$

$$B_s(z) = A_s(z) + A_{s-1}'(z), \tag{5·3}$$

the arbitrary constants in (5·2) *being subject to the condition that there exists a function $\phi(u)$ of u alone, with the asymptotic expansion*

$$\phi(u) \sim 1 + \sum_{s=1}^{\infty} \frac{A_s(c)}{u^s}, \tag{5·4}$$

as $u \to \infty$, for a fixed point c in $\mathbf{D}'$.

Then for large u the equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \{u^2 + f(z)\}\, w \tag{5·5}$$

has solutions $W_1(z)$ and $W_2(z)$ such that if z lies in $\mathbf{D}_1$

$$W_1(z) = \mathrm{e}^{uz}\left\{\sum_{s=0}^{m-1} \frac{A_s(z)}{u^s} + O\left(\frac{1}{u^m}\right)\right\}, \quad W_1'(z) = u\,\mathrm{e}^{uz}\left\{\sum_{s=0}^{m-1} \frac{B_s(z)}{u^s} + O\left(\frac{1}{u^m}\right)\right\}, \tag{5·6}$$

and if z lies in $\mathbf{D}_2$

$$W_2(z) = \mathrm{e}^{-uz}\left\{\sum_{s=0}^{m-1} (-)^s \frac{A_s(z)}{u^s} + O\left(\frac{1}{u^m}\right)\right\}, \quad W_2'(z) = -u\,\mathrm{e}^{-uz}\left\{\sum_{s=0}^{m-1} (-)^s \frac{B_s(z)}{u^s} + O\left(\frac{1}{u^m}\right)\right\}, \tag{5·7}$$

where each of the O's is uniform with respect to z. Here m is an arbitrary unbounded integer, but the solutions $W_1(z)$ and $W_2(z)$ themselves are independent of m.

Typical domains are illustrated in figures 3 and 4. The thickened lines in figure 3 denote cuts and are the boundaries of $\mathbf{D}$; the broken lines are the boundaries of $\mathbf{D}'$. The domain $\mathbf{D}_1$ is the unshaded region in figure 4.

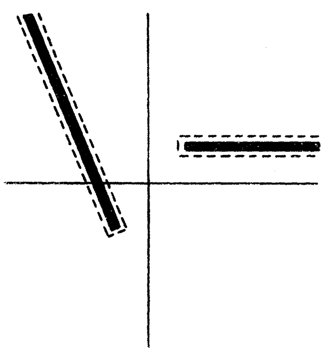

FIGURE 3. Domains $\mathbf{D}$, $\mathbf{D}'$.

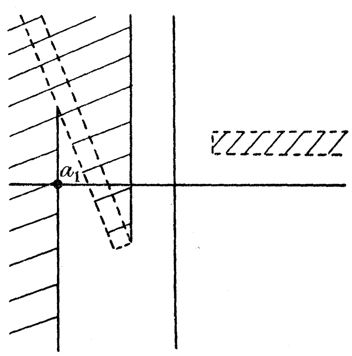

FIGURE 4. Case A: $\mathbf{D}_1$.

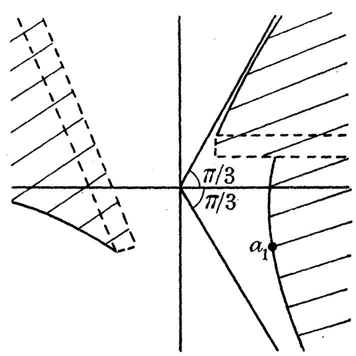

FIGURE 5. Case B: $\mathbf{D}_1$.

Before stating the second theorem we introduce the following notation and definitions. It is supposed now that $z = 0$ is an interior point of the domain $\mathbf{D}'$:

(i) $v \equiv u^{\frac{1}{3}}$; $\rho_1 \equiv 1$, $\rho_2 \equiv \mathrm{e}^{\frac{2}{3}\pi \mathrm{i}}$, $\rho_3 \equiv \mathrm{e}^{-\frac{2}{3}\pi \mathrm{i}}$.

† Here A_s corresponds to A_s and B_s of § 3, B_s to C_s and D_s of § 3, and u has been replaced by u^2.

(ii) a_j $(j = 1, 2, 3)$ is an arbitrary fixed point of the region common to the domain $\mathbf{D}'$ and the sector $\mathbf{S}_j$ (figure 1). If this common region is unbounded then a_j can be the point at infinity on a line lying in it.

(iii) $\mathbf{D}_j$ is the domain comprising those points z of $\mathbf{D}'$ for which

$$|\exp\{-\tfrac{2}{3}(\rho_j z)^{\frac{3}{2}}\}| \geqslant |\exp\{-\tfrac{2}{3}(\rho_j a_j)^{\frac{3}{2}}\}|,$$

and a contour can be found joining z and a_j which lies in $\mathbf{D}'$ and at most two of the sectors $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, and does not cross the level curve of $\exp\{-\tfrac{2}{3}(\rho_j z)^{\frac{3}{2}}\}$ through z.

A typical domain $\mathbf{D}_1$ corresponding to domains $\mathbf{D}$ and $\mathbf{D}'$ of figure 3 is the unshaded region in figure 5. The curved boundaries are level curves of $\exp(-\tfrac{2}{3}z^{\frac{3}{2}})$ (cf. figure 2).

THEOREM B. *Let sequences of functions $A_s(z)$, $B_s(z)$, $C_s(z)$ and $D_s(z)$ be defined by the relations* $A_0(z) = 1$,

$$\left.\begin{aligned} B_s(z) &= \tfrac{1}{2}z^{-\frac{1}{2}}\int_0^z t^{-\frac{1}{2}}\{f(t)\,A_s(t) - A_s''(t)\}\,dt, \\ A_{s+1}(z) &= -\tfrac{1}{2}B_s'(z) + \tfrac{1}{2}\int f(z)\,B_s(z)\,dz + \text{constant}, \end{aligned}\right\} \tag{5·8}$$

$$C_s(z) = A_s'(z) + zB_s(z), \quad D_s(z) = A_s(z) + B_{s-1}'(z), \tag{5·9}$$

the arbitrary constants in the second of (5·8) being subject to the condition that there exists a function $\phi(u)$ having the asymptotic expansion

$$\phi(u) \sim 1 + \sum_{s=1}^{\infty}\frac{A_s(c)}{u^s}, \tag{5·10}$$

as $u\to\infty$, for fixed a point c in $\mathbf{D}'$.

Then the equation

$$\frac{d^2w}{dz^2} = \{uz + f(z)\}\,w \tag{5·11}$$

has solutions $W_j(z)$ $(j = 1, 2, 3)$, such that if z lies in $\mathbf{D}_j$

$$W_j(z) = P_j(vz)\left\{1 + \sum_{s=1}^{m}\frac{A_s(z)}{u^s}\right\} + \frac{P_j'(vz)}{v^2}\sum_{s=0}^{m-1}\frac{B_s(z)}{u^s} + \frac{\exp\{-\tfrac{2}{3}(\rho_j vz)^{\frac{3}{2}}\}}{1+|vz|^{\frac{1}{4}}}\,O\left(\frac{1}{u^{m+\frac{1}{2}}}\right), \tag{5·12}$$

$$W_j'(z) = P_j(vz)\sum_{s=0}^{m-1}\frac{C_s(z)}{u^s} + vP_j'(vz)\left\{1 + \sum_{s=1}^{m}\frac{D_s(z)}{u^s}\right\} + (1+|vz|^{\frac{1}{4}})\exp\{-\tfrac{2}{3}(\rho_j vz)^{\frac{3}{2}}\}\,O\left(\frac{1}{u^m}\right), \tag{5·13}$$

as $u\to\infty$, where the functions $P_j(z)$ $(j = 1, 2, 3)$ are defined by (4·6) and the O's are uniform with respect to z. Here m is an arbitrary unbounded integer, and the solutions $W_j(z)$ are independent of m.

Remarks on the theorems. (i) It is evident that (5·6) and (5·7) are asymptotic expansions in the classical sense of Poincaré. Similarly, it may be verified that (5·12) and (5·13) have the equivalent forms

$$W_j(z) \sim P_j(vz)\sum_{s=0}^{\infty}\frac{A_s(z)}{u^s} + \frac{P_j'(vz)}{v^2}\sum_{s=0}^{\infty}\frac{B_s(z)}{u^s}, \tag{5·14}$$

$$W_j'(z) \sim P_j(vz)\sum_{s=0}^{\infty}\frac{C_s(z)}{u^s} + vP_j'(vz)\sum_{s=0}^{\infty}\frac{D_s(z)}{u^s}. \tag{5·15}$$

(ii) The condition that a function $\phi(u)$ exists having the properties (5·4) or (5·10) is not very restrictive. We could, for example, take any set of values for $A_s(c)$ such that $\Sigma u^{-s}A_s(c)$ converges for large u, or even take $A_{s+1}(c) = 0$ $(s \geqslant 0)$.

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6. Application of the theorems

The coefficients $A_s(z)$, $B_s(z)$, $C_s(z)$ and $D_s(z)$ defined by equations (5·2), (5·3), (5·8) and (5·9) are regular functions of z in **D**. In these relations they are expressed in the form of integrals, and it may not always be convenient or even possible to perform the integrations analytically and obtain explicit expressions for the coefficients. The integrals may usually be evaluated numerically, however, and tables of coefficients can be prepared for the standard solutions of a given differential equation. An important practical point is that each quantity occurring in the series (5·6), (5·7), (5·14) and (5·15) is a function of a *single* variable, and the general tabulation of the solutions $W_j(z)$, which themselves are functions of two variables z and u, is mainly reduced to the preparation of a few single-entry tables. Such tables have, in fact, already been compiled by the writer for Bessel's and Legendre's equations.

These points are illustrated in the paper immediately following this, in which theorems A and B are used to obtain asymptotic expansions of Bessel functions of large order. In addition, it is shown there how the asymptotic series may be reverted to obtain powerful asymptotic expansions for the zeros and stationary values of the functions.

7. Previous results

The earliest writings on the asymptotic solution of differential equations for large values of a parameter appear to be those of Carlini,† Green (1837) and Liouville (1837). These authors gave, without rigorous investigation, formal asymptotic solutions in intervals free from transition points (defined in § 1). Carlini considered a form of Bessel's equation for large orders, but the equations of Green and Liouville were more general.

Extensive researches have been published on the existence theory of asymptotic solutions in intervals free from transition points. The principal contributors are Horn (1899), Schlesinger (1907), Birkhoff (1908), Tamarkin (1928) and Turrittin (1936). These writers investigated the asymptotic solution of an arbitrary number of simultaneous first-order differential equations for real values of the independent variable z and large complex values of the parameter u. Theorem A, on the other hand (in the case of second-order equations), provides a satisfactory account of the behaviour of solutions in the complex z-plane. There is a further marked difference. The relevant final result of the earlier writers applied to equation (5·5) shows that for each value of m, pairs of solutions exist having the asymptotic forms (5·6) and (5·7). Theorem A gives the more powerful result that there exists a pair of solutions having the properties (5·6) and (5·7) for *all* values of m; in a sense this means we can take $m = \infty$ in the earlier result.

If the range of integration extends from‡ $z = -\infty$ to $z = +\infty$ it can be deduced quite simply from the earlier result that solutions $W_1^*(z)$ and $W_2^*(z)$ exist, independent of m, having the asymptotic forms (5·6) and (5·7) with the arbitrary constants in (5·2) fixed by

$$A_{s+1}(-\infty) = 0 \text{ for } W_1^*(z), \quad A_{s+1}(+\infty) = 0 \text{ for } W_2^*(z) \quad (s \geq 0). \tag{7·1}$$

For let $W_{m,1}(z)$ and $W_{m,2}(z)$ be the solutions having the properties

$$W_{m,1}(z) = e^{uz}\left\{1+\sum_{s=1}^{m-1}\frac{A_s(z)}{u^s}+O\left(\frac{1}{u^m}\right)\right\}, \quad W_{m,2}(z) = e^{-uz}\left\{1+\sum_{s=1}^{m-1}(-)^s\frac{A_s(z)}{u^s}+O\left(\frac{1}{u^m}\right)\right\}, \tag{7·2}$$

† An account of Carlini's paper is given by Watson (1944, p. 6).
‡ There are complications if the range of integration is finite.

with the values of the coefficients $A_s(z)$ fixed by the first of (7·1). We define $W_1^*(z)$ to be the solution of (5·5) with the condition†

$$W_1^*(z) \sim e^{uz} \quad \text{as} \quad z \to -\infty. \tag{7·3}$$

It can be expressed in the form

$$W_1^*(z) = k_{m,1} W_{m,1}(z) + k_{m,2} W_{m,2}(z), \tag{7·4}$$

where $k_{m,1}$ and $k_{m,2}$ are independent of z. Letting $z \to -\infty$, we see from (7·2) that for all sufficiently large u

$$k_{m,2} = 0, \quad k_{m,1} = 1 + O(u^{-m}).$$

Substituting this result and (7·2) in (7·4), we obtain

$$W_1^*(z) = e^{uz}\left\{1 + \sum_{s=1}^{m-1} \frac{A_s(z)}{u^s} + O\left(\frac{1}{u^m}\right)\right\}, \tag{7·5}$$

and, since $W_1^*(z)$ is obviously independent of m, this is the result stated. Similarly for $W_2^*(z)$.

If c is any point of the real axis, we may multiply the solution $W_1^*(z)$ by $1/\omega(u)$, where $\omega(u)$ is a function of u alone having the Poincaré asymptotic expansion

$$\omega(u) \sim 1 + \frac{A_1(c)}{u} + \frac{A_2(c)}{u^2} + \ldots, \tag{7·6}$$

and the coefficients A_s in this relation are fixed by the first of (7·1). It would then follow that a solution $W_1(z)$, independent of m, exists having the property (5·6) with a new set of coefficients A_s; the arbitrary constants in (5·2) are now determined by

$$A_{s+1}(c) = 0 \quad (s \geqslant 0).$$

This is essentially the result given by theorem A in the case of real variables. The proof is incomplete, however, until it is established that $\omega(u)$ *does* exist having the property (7·6). This particular result is in fact obtained later in case B (equation (10·3) with $z = c$), and a similar proof of it holds in case A. The proofs of theorems A and B given in this paper do not follow these lines however, and are quite independent of earlier results.

One of the first successful attempts to deal with the case when there is a transition point in the interval of integration‡ is that of Jeffreys§ (1924; or Jeffreys & Jeffreys 1946, pp. 492–4). He reasoned that if the function $p(z)$ in (1·1) has a simple zero at $z = z_0$, then near this point the differential equation is approximately represented by

$$\frac{d^2w}{dz^2} = up'(z_0)\,(z - z_0)\,w,$$

solutions of which are Ai (Z) and Bi (Z), where

$$Z \equiv \{up'(z_0)\}^{\frac{1}{3}}\,(z - z_0).$$

The drawback to this procedure is that the functions Ai (Z) and Bi (Z) are reasonable approximations to solutions of (1·1) only near $z = z_0$, and in any case there is no indication of the error of the approximation.

† That such a solution exists follows from (5·1) and the ordinary asymptotic theory of linear differential equations (Ince 1944, §7·31).

‡ A review of the literature concerned with this case has been published recently by the California Institute of Technology (1953).

§ In a recent paper (1953) Jeffreys points out that this approach was partially anticipated by Gans and Rayleigh.

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These disadvantages are largely removed in the method originally given by Langer (1931). He showed how an equation of the form (1·1) may be transformed, as shown in § 2, into (2·8), and he proved that under certain conditions equation (2·8) possesses solutions which for large u are asymptotic to Bessel functions of order one-third. The interval in which these approximations are valid is actually a function of u which shrinks to zero as $u \to \infty$. However, Langer also gave approximations to the same solutions in terms of exponential functions which are valid outside this interval. He later extended these results to the case of complex variables (1932) and to different kinds of transition points (1934, 1935).

The asymptotic form of solution (5·12) was also given, in effect, by Langer (1937, 1949), who proved a result concerning its validity closely analogous to that of Birkhoff and others quoted above for case A; it concerns only real values of the independent variable, and the asymptotic solutions obtained depend on the integer m. An investigation of the solutions when the independent variable is complex has been made by Cherry (1950), who has proved a result resembling theorem B of § 5.

Using the present notation, Cherry shows that solutions $W_{m,j}(z)$ $(j = 1, 2, 3)$ of (5·11) exist for large positive u such that

$$W_{m,j}(z) = P_j(vz)\left\{\sum_{s=0}^{m-1}\frac{A_s(z)}{u^s}+\frac{a_m(z,u)}{u^m}\right\}+\frac{P_j'(vz)}{v^2}\left\{\sum_{s=0}^{m-1}\frac{B_s(z)}{u^s}+\frac{b_m(z,u)}{u^m}\right\},$$

where $|a_m(z,u)|$ and $|b_m(z,u)|$ are bounded; this result is similar to that of Birkhoff for real variables quoted above for case A (see equations (7·2)). By making an additional assumption Cherry then deduces that in certain bounded z-domains solutions $W_j^*(z)$ exist, independent of m, having the asymptotic expansions

$$W_j^*(z) \sim P_j(vz)\sum_{s=0}^{\infty}\frac{A_s^j(z)}{u^s}+\frac{P_j'(vz)}{v^2}\sum_{s=0}^{\infty}\frac{B_s^j(z)}{u^s}, \qquad (7·7)$$

as $u \to \infty$, uniformly with respect to z, where the coefficients $A_s^j(z)$, $B_s^j(z)$ are the values of $A_s(z)$, $B_s(z)$ given by (5·8) with the arbitrary constants fixed by

$$A_{s+1}^j(\infty\,\rho_j^{-1}) = 0 \quad (s \geqslant 0),$$

where ρ_j is defined in § 5. The assumption is that $f(z)$ is regular in sectors of non-zero angles extending to infinity centred on the rays $\arg z = 0, \pm\frac{2}{3}\pi$, and in these sectors $f(z) = O(|z|)$ as $|z| \to \infty$ (Cherry 1950, p. 232). The result (7·7) is similar to (7·5) above for case A.

Unlike theorem B Cherry's result does not show that solutions exist when the values of $A_1(z)$, $A_2(z)$, ..., in (5·8) are prescribed at arbitrary points. Moreover, in this theorem it is not stipulated that the rays $\arg z = 0, \pm\frac{2}{3}\pi$, are included in the domain in which $f(z)$ is regular, or that the domain of uniform validity of the expansion has to be bounded.

The rest of this paper is concerned with the proof of the theorems of § 5. The methods used differ considerably from those of Langer and Cherry.

8. Preliminaries in the proof of the existence theorems

Only a proof for theorem B will be given, since a similar but simpler proof holds for theorem A.

In the next section we consider properties of the coefficients $A_s(z)$ and $B_s(z)$ defined by (5·8), and it is convenient to introduce here some definitions concerning the function $f(z)$.

It was postulated in § 5 that $f(z)$ is regular in a simply-connected open domain **D**, the boundaries of which consist of a finite number of straight lines, and that if **D** is unbounded, $f(z) = O(|z|^{-2})$ as $|z| \to \infty$, uniformly with respect to $\arg z$. Consequently in **D**,

$$|f(z)| \leqslant F_1, \quad |z^2 f(z)| \leqslant F_2, \tag{8·1}$$

where F_1 and F_2 are assignable constants. We shall sometimes use the symbol F generically to denote a combination of these constants.

From Liouville's theorem it follows that **D** cannot be the whole z-plane, unless $f(z) \equiv 0$. In other words, a definite boundary always exists.

Auxiliary domains. Let us suppose that straight lines are drawn in **D** parallel to the boundaries and at a distance δ from them; if the boundaries include cuts then semicircles of radius δ are described centred on the end-points. Clearly a constant d can be assigned such that for every positive $\delta \leqslant d$, the aggregate of lines and semicircles defines the boundary of another simply-connected domain which also contains $z = 0$ as an interior point. This domain will be denoted by $\mathbf{D}_\delta$, and when $\delta = d$ by $\mathbf{D}_d$.

Neighbourhoods of infinity. There is only a finite number of points at which the boundaries of the domains **D** and $\mathbf{D}_d$ change direction abruptly, including the end-points of cuts. Let the one most remote from the origin be of affix R', and let $R \equiv \max(R'+d, 1)$. The simply-connected subdomains of **D** lying outside the circle $|z| = R$ will be denoted by $\mathbf{E}^1, \mathbf{E}^2, \ldots$, or, typically, by **E**. Similarly, the simply-connected subdomains of $\mathbf{D}_\delta$ lying outside the circle $|z| = R+\delta$ will be denoted by $\mathbf{E}^1_\delta, \mathbf{E}^2_\delta, \ldots$, or typically by $\mathbf{E}_\delta$.

Where the symbol δ is used in the following sections it is assumed that

$$0 < \delta \leqslant \min(d, \tfrac{1}{2}R, 1). \tag{8·2}$$

9. Properties of the coefficients $A_s(z)$ and $B_s(z)$

These coefficients are defined by $A_0(z) = 1$ and (see (5·8))

$$B_s(z) = \tfrac{1}{2}z^{-\frac{1}{2}} \int_0^z t^{-\frac{1}{2}} \{f(t)\, A_s(t) - A_s''(t)\}\, \mathrm{d}t \quad (s \geqslant 0), \tag{9·1}$$

$$A_{s+1}(z) = -\tfrac{1}{2}B_s'(z) + \tfrac{1}{2}\int_0^z f(t)\, B_s(t)\, \mathrm{d}t + \text{constant} \quad (s \geqslant 0). \tag{9·2}$$

The integration paths in (9·1) and (9·2) both lie in **D**, and the branches of the square roots occurring in (9·1) are taken to be the principal ones near the origin and defined by continuity elsewhere. Clearly $A_s(z)$ and $B_s(z)$ are regular functions of z in **D**.

The arbitrary constant in (9·2) is fixed temporarily by the condition

$$A_{s+1}(c) = 0 \quad (s \geqslant 0), \tag{9·3}$$

for some point c, which may be at infinity, in $\mathbf{D}_d$. Then

$$A_{s+1}(z) = -\tfrac{1}{2}B_s'(z) + \tfrac{1}{2}\int_c^z f(t)\, B_s(t)\, \mathrm{d}t + \tfrac{1}{2}B_s'(c). \tag{9·4}$$

This restriction is relaxed in § 11, and in any case it is not necessary for the truth of lemma 1 which follows.

Lemma 1. *If* $|z|\to\infty$ *in* $\mathbf{E}_\delta$, *then*

$$A_s(z) = \alpha_s + O(|z|^{-\frac{3}{2}}), \tag{9·5}$$

$$B_s(z) = \beta_s z^{-\frac{1}{2}} + O(|z|^{-\frac{3}{2}}), \tag{9·6}$$

uniformly with respect to $\arg z$. *The quantities* α_s *and* β_s *are constants, and the branch of* $z^{-\frac{1}{2}}$ *in* (9·6) *is continuous in* $\mathbf{E}_\delta$.

This may be proved by induction. Suppose that (9·5) is true for a particular s as $|z|\to\infty$ in $\mathbf{E}_\eta$, where η is an arbitrary number lying in the range $0<\eta<\delta$. Then in $\mathbf{E}_\eta$

$$A_s(z) = \alpha_s + a_s(z), \quad \text{where} \quad |a_s(z)| < \lambda_s |z|^{-\frac{3}{2}}, \tag{9·7}$$

λ_s being a constant.

Now if z lies in $\mathbf{E}_\epsilon$, where $\eta<\epsilon\leqslant\delta$, we have from Cauchy's integral formula

$$A_s''(z) = \frac{1}{\pi\mathrm{i}}\int_{|t-z|=\epsilon-\eta} \frac{A_s(t)}{(t-z)^3}\,\mathrm{d}t = \frac{1}{\pi\mathrm{i}}\int_{|t-z|=\epsilon-\eta} \frac{a_s(t)}{(t-z)^3}\,\mathrm{d}t,$$

and so, using (9·7),

$$|A_s''(z)| < \frac{2}{(\epsilon-\eta)^2}\frac{\lambda_s}{\{|z|-(\epsilon-\eta)\}^{\frac{3}{2}}} = O(|z|^{-\frac{3}{2}}), \quad \text{as} \quad |z|\to\infty. \tag{9·8}$$

From this result and (5·1), (9·7) it is seen that the integral

$$\int_0^z t^{-\frac{1}{2}}\{f(t)\,A_s(t) - A_s''(t)\}\,\mathrm{d}t$$

converges as $|z|\to\infty$ in $\mathbf{E}_\epsilon$, and hence (9·1) may be written in the form

$$B_s(z) = \beta_s z^{-\frac{1}{2}} + b_s(z), \quad b_s(z) = -\tfrac{1}{2}z^{-\frac{1}{2}}\int_z^\infty t^{-\frac{1}{2}}\{f(t)\,A_s(t) - A_s''(t)\}\,\mathrm{d}t, \tag{9·9}$$

where β_s is a constant, and the upper limit ∞ denotes a point at infinity in $\mathbf{E}_\epsilon$. From (5·1), (9·7) and (9·8) we see that

$$b_s(z) = O(|z|^{-\frac{3}{2}}), \tag{9·10}$$

as $|z|\to\infty$ in $\mathbf{E}_\epsilon$. Thus if (9·5) is true in $\mathbf{E}_\eta$, then (9·6) holds in $\mathbf{E}_\epsilon$.

Next, it follows from (5·1), (9·10) and the first of (9·9) that $\int_0^z f(t)\,B_s(t)\,\mathrm{d}t$ converges as $|z|\to\infty$ in $\mathbf{E}_\epsilon$. We may therefore write (9·2) in the form

$$A_{s+1}(z) = \alpha_{s+1} + a_{s+1}(z),$$

where α_{s+1} is a constant and

$$a_{s+1}(z) = -\tfrac{1}{2}B_s'(z) - \tfrac{1}{2}\int_z^\infty f(t)\,B_s(t)\,\mathrm{d}t. \tag{9·11}$$

From (9·1) we obtain

$$B_s'(z) = \{f(z)\,A_s(z) - A_s''(z) - B_s(z)\}/(2z), \tag{9·12}$$

(cf. (3·9)), and referring to (9·5), (9·8), (9·9) and (9·10) it is seen that $B_s'(z) = O(|z|^{-\frac{3}{2}})$. Hence from (9·11) we deduce that $a_{s+1}(z) = O(|z|^{-\frac{3}{2}})$, as $|z|\to\infty$ in $\mathbf{E}_\epsilon$. This completes the proof of lemma 1.

Lemma 2. *When* z *lies in* $\mathbf{D}_\delta$ *and* $|t|\leqslant T$, *the series*

$$A(z,t) = \sum_{s=0}^\infty A_s(z)\frac{t^{2s}}{(2s)!}, \quad B(z,t) = \sum_{s=0}^\infty B_s(z)\frac{t^{2s}}{(2s)!}, \tag{9·13}$$

converge uniformly with respect to z *and* t. *Here* $T>0$ *is an assignable constant which may depend on* δ, *but is independent of* z.

Let the maximum† moduli in $\mathbf{D}_\delta$ of $A_s(z)$, $A_s'(z)$, $A_s''(z)$, $B_s(z)$ and $B_s'(z)$ be denoted by $M_s(\delta)$, $M_s^{(1)}(\delta)$, $M_s^{(2)}(\delta)$, $P_s(\delta)$ and $P_s^{(1)}(\delta)$ respectively, and let η, ϵ be arbitrary numbers such that $0<\eta<\epsilon\leqslant\delta$.

From (9·1) we obtain

$$|B_s(z)|\leqslant\tfrac{1}{2}|z|^{-\frac{1}{2}}\left|\int_0^z t^{-\frac{1}{2}}\{f(t)\,A_s(t)-A_s''(t)\}\,\mathrm{d}t\right|.$$

If the domain $\mathbf{D}_\epsilon$ were a star-domain the path of integration could be taken to be a straight line, and using (8·1) we would immediately deduce that

$$P_s(\epsilon)\leqslant F_1M_s(\epsilon)+M_s^{(2)}(\epsilon).$$

In general, however, the path cannot be taken to be a single straight line but by considering separately the three parts into which it is divided by the circles $|z|=R$ and $|z|=\rho$, where ρ is the shortest distance between $z=0$ and the boundary of $\mathbf{D}_d$, it may be seen that

$$P_s(\epsilon)\leqslant\kappa\{F_1M_s(\epsilon)+M_s^{(2)}(\epsilon)\},\tag{9·14}$$

where $\kappa\geqslant1$ is a constant whose value depends on the configuration of the boundaries of the domains $\mathbf{D}$ and $\mathbf{D}_d$.

Similarly, from (9·4) we obtain, using now the symbol κ generically,

$$M_{s+1}(\epsilon)\leqslant P_s^{(1)}(\epsilon)+\kappa FP_s(\epsilon).\tag{9·15}$$

From the maximum-modulus theorem and lemma 1 with δ replaced by ϵ, it follows that $|B_s(z)|$ attains its maximum value in $\mathbf{D}_\epsilon$ on the boundary. Hence from (8·1) and (9·12) we find that

$$P_s^{(1)}(\epsilon)<\{F_1M_s(\epsilon)+M_s^{(2)}(\epsilon)+P_s(\epsilon)\}/(2\rho).\tag{9·16}$$

Combining (9·14), (9·15) and (9·16), we obtain

$$M_{s+1}(\epsilon)<\{\kappa^2F+(1+\kappa)\,(2\rho)^{-1}\}\{F_1M_s(\epsilon)+M_s^{(2)}(\epsilon)\}.\tag{9·17}$$

Now if z lies in $\mathbf{D}_\epsilon$, we have from Cauchy's integral formula

$$A_s''(z)=\frac{1}{\pi\mathrm{i}}\int_{|t-z|=\epsilon-\eta}\frac{A_s(t)}{(t-z)^3}\,\mathrm{d}t,$$

and hence

$$M_s^{(2)}(\epsilon)\leqslant2(\epsilon-\eta)^{-2}\,M_s(\eta).\tag{9·18}$$

Substituting this result in (9·17) and using the fact that $M_s(\epsilon)\leqslant M_s(\eta)$, we obtain

$$\left.\begin{aligned}M_{s+1}(\epsilon)&<GM_s(\eta),\\ \text{where}\quad G&=\{\kappa^2F+(1+\kappa)\,(2\rho)^{-1}\}\{F_1+2(\epsilon-\eta)^{-2}\}.\end{aligned}\right\}\tag{9·19}$$

This result is the key to the proof, and it may now be completed as follows. Let us write

$$\delta_{n,s}\equiv\frac{s-n}{s}\delta\quad(n=0,1,2,\ldots,s).\tag{9·20}$$

Then from (9·19) we obtain

$$M_s(\delta)\equiv M_s(\delta_{0,s})<G_sM_{s-1}(\delta_{1,s})<G_s^2M_{s-2}(\delta_{2,s})<\ldots<G_s^s,\tag{9·21}$$

$$\left.\begin{aligned}\text{where}\quad G_s&\equiv\{\kappa^2F+(1+\kappa)\,(2\rho)^{-1}\}\,(F_1+2s^2\delta^{-2})\leqslant k_\delta s^2\quad(s\geqslant1),\\ k_\delta&\equiv\{\kappa^2F+(1+\kappa)\,(2\rho)^{-1}\}\,(F_1+2\delta^{-2}).\end{aligned}\right\}\tag{9·22}$$

† The existence of these maxima follows from lemma 1.

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Thus if z lies in $\mathbf{D}_\delta$, then

$$|A_s(z)| \leqslant M_s(\delta) < k_\delta^s s^{2s} \quad (s \geqslant 1). \tag{9·23}$$

Further, from (9·18) to (9·22) we deduce that

$$M_{s-1}^{(2)}(\delta) \leqslant 2\delta^{-2}s^2 M_{s-1}(\delta_{1,s}) < 2\delta^{-2}s^2 G_s^{s-1} \leqslant 2\delta^{-2}k_\delta^{s-1}s^{2s}. \tag{9·24}$$

Substituting (9·23) and (9·24) in (9·14), we see that if z lies in $\mathbf{D}_\delta$, then

$$|B_s(z)| \leqslant P_s(\delta) < \kappa k_\delta^s\{F_1 s^{2s} + 2\delta^{-2}(s+1)^{2s+2}\}. \tag{9·25}$$

The truth of lemma 2 is now obvious from the results (9·23), (9·25) and Stirling's formula

$$(2s)! \sim \surd(2\pi)\, s^{2s+\frac{1}{2}}\, 2^{2s+\frac{1}{2}}\, \mathrm{e}^{-2s}. \tag{9·26}$$

The value of T can be taken to be any positive number less than $2\,\mathrm{e}^{-1}k_\delta^{-\frac{1}{2}}$.

Lemma 3. *If z lies in $\mathbf{E}_\delta$ and $|t| \leqslant \tau$, then*

$$\frac{\partial}{\partial z}A(z,t) \equiv \sum_{s=1}^{\infty} A_s'(z)\frac{t^{2s}}{(2s)!} = O(|z|^{-\frac{3}{2}}), \tag{9·27}$$

$$\frac{\partial^2}{\partial z\,\partial t}A(z,t) \equiv \sum_{s=1}^{\infty} A_s'(z)\frac{t^{2s-1}}{(2s-1)!} = O(|z|^{-\frac{3}{2}}), \tag{9·28}$$

as $|z| \to \infty$, uniformly with respect to t and $\arg z$. Here τ is an assignable number independent of z, and $0 < \tau \leqslant T$.

To prove this result it is necessary to examine how the constants implied in the O's occurring in the proof of lemma 1 depend on s.

As in the proof of lemmas 1 and 2 it is supposed that η and ϵ are arbitrary numbers such that $0 < \eta < \epsilon \leqslant \delta$. We first investigate the connexion between λ_s and λ_{s+1}, where λ_s is the constant introduced in (9·7).

If z lies in $\mathbf{E}_\epsilon$, then from (9·8)

$$|A_s''(z)| < \frac{2}{(\epsilon-\eta)^2}\frac{\lambda_s}{\{|z|-(\epsilon-\eta)\}^{\frac{3}{2}}} < \frac{2^{\frac{5}{2}}\lambda_s}{(\epsilon-\eta)^2|z|^{\frac{3}{2}}}, \tag{9·29}$$

since from (8·2), $\quad |z|-(\epsilon-\eta) > |z|-\delta > |z|(1-\delta R^{-1}) > \frac{1}{2}|z|.$

Using this result and (8·1), (9·7), we find that

$$|z^{-\frac{1}{2}}\{f(z)A_s(z) - A_s''(z)\}| < \frac{F_2}{|z|^{\frac{3}{2}}}\left(|\alpha_s| + \frac{\lambda_s}{|z|^{\frac{1}{2}}}\right) + \frac{2^{\frac{5}{2}}\lambda_s}{(\epsilon-\eta)^2|z|^2} < \frac{\mu_s}{|z|^2}, \tag{9·30}$$

where
$$\mu_s = F_2(|\alpha_s| + \lambda_s) + 2^{\frac{5}{2}}(\epsilon-\eta)^{-2}\lambda_s, \tag{9·31}$$
since $|z| > 1$ in $\mathbf{E}_\epsilon$ (see § 8).

In the integral (9·9) the path of integration may be taken to be a straight line whose equation is of the form $t = z + p\,\mathrm{e}^{\mathrm{i}\varpi}$, where $0 \leqslant p \leqslant \infty$, ϖ is a constant, and on the path $|t|^2 \geqslant |z|^2 + p^2$. Accordingly, we obtain from (9·9) and (9·30)

$$|b_s(z)| < \frac{1}{2|z|^{\frac{1}{2}}}\int_0^\infty \frac{\mu_s\,\mathrm{d}p}{|z|^2+p^2} = \frac{\pi\mu_s}{4|z|^{\frac{3}{2}}}. \tag{9·32}$$

Substituting this result and (9·7), (9·9) and (9·29) in (9·12), we obtain

$$|B_s'(z)| < \frac{1}{2|z|}\left\{\frac{F_2}{|z|^2}\left(|\alpha_s| + \frac{\lambda_s}{|z|^{\frac{1}{2}}}\right) + \frac{2^{\frac{5}{2}}\lambda_s}{(\epsilon-\eta)^2|z|^{\frac{3}{2}}} + \frac{|\beta_s|}{|z|^{\frac{1}{2}}} + \frac{\pi\mu_s}{4|z|^{\frac{3}{2}}}\right\} < \frac{|\beta_s| + 2\mu_s}{2|z|^{\frac{3}{2}}}. \tag{9·33}$$

Similarly, from (9·11) using (9·32) and (9·33), we find that

$$|a_{s+1}(z)| < \frac{|\beta_s|+2\mu_s}{4|z|^{\frac{3}{2}}} + \frac{1}{2}\int_0^\infty \frac{F_2(|\beta_s|+\mu_s)}{(|z|^2+p^2)^{\frac{3}{4}}}\,\mathrm{d}p < (F_2+\tfrac{1}{2})(|\beta_s|+\mu_s)|z|^{-\frac{1}{2}}.$$

Substituting for μ_s from (9·31) and using (8·2), we prove that:

$$\left.\begin{aligned} &\textit{If} && |a_s(z)| < \lambda_s |z|^{-\frac{1}{2}} \quad \textit{in} \quad \mathbf{E}_\eta, \\ &\textit{then} && |a_{s+1}(z)| < F\{|\alpha_s|+|\beta_s|+\lambda_s(\epsilon-\eta)^{-2}\}|z|^{-\frac{1}{2}} \quad \textit{in} \quad \mathbf{E}_\epsilon. \end{aligned}\right\} \tag{9·34}$$

We now consider the magnitudes of $|\alpha_s|$ and $|\beta_s|$. Letting $|z|\to\infty$ in $\mathbf{E}_\delta$, we see from (9·7) and (9·23) that

$$|\alpha_s| < k_\delta^s s^{2s}, \tag{9·35}$$

where k_δ is defined by (9·22). Next we have by definition (see (9·1) and (9·6))

$$\beta_s = \frac{1}{2}\int_0^\infty t^{-\frac{1}{2}}\{f(t)\,A_s(t) - A_s''(t)\}\,\mathrm{d}t. \tag{9·36}$$

If t lies in $\mathbf{E}_\epsilon$, then from (9·29)

$$|A_s''(t)| < 2^{\frac{3}{2}}\lambda_s(\epsilon-\eta)^{-2}|t|^{-\frac{3}{2}},$$

and if t lies anywhere in $\mathbf{D}_\epsilon$, then clearly

$$|A_s(t)| \leqslant M_s(\epsilon), \quad |A_s''(t)| \leqslant M_s^{(2)}(\epsilon).$$

Substituting these estimates in (9·36), we see that

$$|\beta_s| < \kappa F\{M_s(\epsilon) + M_s^{(2)}(\epsilon) + \lambda_s(\epsilon-\eta)^{-2}\},$$

where κ is the generic constant introduced in (9·14) and (9·15). Hence, with the aid of (9·23) and (9·24), we obtain

$$|\beta_s| < \kappa F\epsilon^{-2}\{k_\epsilon^s(s+1)^{2s+2} + \lambda_s(\epsilon-\eta)^{-2}\}, \tag{9·37}$$

where k_ϵ is defined by (9·22) with δ replaced by ϵ. Substituting (9·35) and (9·37) in (9·34) and changing s into $n-1$, we see that:

$$\left.\begin{aligned} &\textit{If} && |z^{\frac{1}{2}}a_{n-1}(z)| < \lambda_{n-1} \quad \textit{in} \quad \mathbf{E}_\eta, \\ &\textit{then} && |z^{\frac{1}{2}}a_n(z)| < \kappa F\epsilon^{-2}\{k_\epsilon^{n-1}n^{2n} + \lambda_{n-1}(\epsilon-\eta)^{-2}\} \quad \textit{in} \quad \mathbf{E}_\epsilon. \end{aligned}\right\} \tag{9·38}$$

Let ζ be any fixed number in the range $0<\zeta<\delta$, and let us write

$$\zeta_{n,s} \equiv \zeta + \frac{n}{s}(\delta-\zeta) \quad (n = 0, 1, 2, \ldots, s;\ s \geqslant 1).$$

Then taking $\eta = \zeta_{0,s} = \zeta$, $\epsilon = \zeta_{1,s}$ and $n = 1$ in the result (9·38), we see that

$$|z^{\frac{1}{2}}a_1(z)| < \kappa F\zeta_{1,s}^{-2} < \kappa F\zeta^{-2} = \lambda_{1,s},$$

say. Similarly,

$$|z^{\frac{1}{2}}a_2(z)| < \kappa F\zeta_{2,s}^{-2}\{k_\epsilon 2^4 + \lambda_{1,s}s^2(\delta-\zeta)^{-2}\} < \kappa F\zeta^{-2}\{k_\zeta 2^4 + \lambda_{1,s}s^2(\delta-\zeta)^{-2}\} = \lambda_{2,s},$$

say, and generally,

$$|z^{\frac{1}{2}}a_n(z)| < \lambda_{n,s} \quad (n = 1, 2, \ldots, s), \tag{9·39}$$

where

$$\lambda_{n,s} = \kappa F\zeta^{-2}\{k_\zeta^{n-1}n^{2n} + \lambda_{n-1,s}s^2(\delta-\zeta)^{-2}\}. \tag{9·40}$$

Let us write

$$\lambda_{n,s}^* \equiv K^n s^{2n+2},$$

where

$$K = \max\,[k_\zeta, \kappa F\zeta^{-2}\{1+(\delta-\zeta)^{-2}\}]. \tag{9·41}$$

Then it is evident that

$$\kappa F\zeta^{-2}\{k_\zeta^{n-1}n^{2n}+\lambda^*_{n-1,s}s^2(\delta-\zeta)^{-2}\}<\kappa F\zeta^{-2}\lambda^*_{n-1,s}\{1+s^2(\delta-\zeta)^{-2}\}<Ks^2\lambda^*_{n-1,s}=\lambda^*_{n,s}.$$

From this inequality and (9·40) it follows by induction that

$$\lambda_{n,s}<\lambda^*_{n,s}\quad (n=1,2,\ldots,s),$$

and, in particular, $\lambda_{s,s}<\lambda^*_{s,s}=K^s s^{2s+2}.$

Collecting these results, we see that if z lies in $\mathbf{E}_\delta$ then

$$|A_s(z)-\alpha_s|<K^s s^{2s+2}|z|^{-\frac{3}{2}},$$

where K is independent of z *and* s. Using Cauchy's integral formula we easily deduce from this result that

$$|A_s'(z)|<\text{constant}\times K^s s^{2s+2}|z|^{-\frac{3}{2}},$$

and from this inequality and (9·26) the truth of lemma 3 is established.

10. Proof of theorem B

Let us suppose that the domain $\mathbf{D}_\delta$ (defined in § 8) contains the whole of $\mathbf{D}'$ (defined in § 5); this can always be arranged by taking δ to be sufficiently small. Consider the functions $L_j(z)$ $(j=1,2,3)$ defined by

$$L_j(z)=u^{\frac{1}{3}}P_j(vz)\int_0^\tau e^{-t\sqrt{u}}A(z,t)\,dt+u^{-\frac{1}{3}}P_j'(vz)\int_0^\tau e^{-t\sqrt{u}}B(z,t)\,dt, \tag{10·1}$$

where $v\equiv u^{\frac{1}{3}}$, $A(z,t)$ and $B(z,t)$ are given by (9·13), and τ is the quantity introduced in lemma 3. More accurately, τ is the least of all such quantities when all the subdomains $\mathbf{E}_\delta^1, \mathbf{E}_\delta^2, \ldots$, are taken into account. The derivative $L_j'(z)$ is given by

$$L_j'(z)=u^{\frac{1}{3}}P_j(vz)\int_0^\tau e^{-t\sqrt{u}}\left(\frac{\partial A}{\partial z}+zB\right)dt+u^{-\frac{1}{3}}P_j'(vz)\int_0^\tau e^{-t\sqrt{u}}\left(uA+\frac{\partial B}{\partial z}\right)dt. \tag{10·2}$$

Substituting the expansions (9·13) for $A(z,t)$ and $B(z,t)$ in (10·1) and integrating term by term, we may verify without difficulty that

$$L_j(z)\sim P_j(vz)\sum_{s=0}^\infty\frac{A_s(z)}{u^s}+\frac{P_j'(vz)}{v^2}\sum_{s=0}^\infty\frac{B_s(z)}{u^s}, \tag{10·3}$$

as $u\to\infty$, uniformly with respect to z in $\mathbf{D}'$. Similarly, using (5·9), we obtain from (10·2)

$$L_j'(z)\sim P_j(vz)\sum_{s=0}^\infty\frac{C_s(z)}{u^s}+vP_j'(vz)\sum_{s=0}^\infty\frac{D_s(z)}{u^s}. \tag{10·4}$$

Comparing (10·3) and (10·4) with (5·14) and (5·15), we see that $L_j(z)$ has the desired asymptotic form. It does not, however, satisfy the differential equation (5·11). For, from (10·1) and (10·2), we derive

$$\frac{d^2L_j}{dz^2}-(uz+f)L_j=u^{\frac{1}{3}}P_j(vz)\int_0^\tau e^{-t\sqrt{u}}\left(\frac{\partial^2A}{\partial z^2}-fA+2z\frac{\partial B}{\partial z}+B\right)dt$$

$$+u^{-\frac{1}{3}}P_j'(vz)\int_0^\tau e^{-t\sqrt{u}}\left(2u\frac{\partial A}{\partial z}+\frac{\partial^2B}{\partial z^2}-fB\right)dt.$$

Now, using (9·13) and (3·9) with $g(z) = z$, we find that

$$\frac{\partial^2 A}{\partial z^2} - fA + 2z\frac{\partial B}{\partial z} + B = 0, \quad 2u\frac{\partial A}{\partial z} + \frac{\partial^2 B}{\partial z^2} - fB = 2u\frac{\partial A}{\partial z} - 2\frac{\partial^3 A}{\partial z\,\partial t^2},$$

and so

$$\frac{\mathrm{d}^2 L_j}{\mathrm{d}z^2} - (uz+f)\,L_j = -P_j'(vz)\,R(z), \tag{10·5}$$

where

$$R(z) = 2u^{-\frac{1}{3}}\int_0^\tau \mathrm{e}^{-t\sqrt{u}}\frac{\partial^3 A}{\partial z\,\partial t^2}\,\mathrm{d}t - 2u^{\frac{2}{3}}\int_0^\tau \mathrm{e}^{-t\sqrt{u}}\frac{\partial A}{\partial z}\,\mathrm{d}t.$$

Integrating the last integral twice by parts, we find

$$R(z) = 2v\,\mathrm{e}^{-\tau\sqrt{u}}\{R_1(z) + u^{-\frac{1}{2}}R_2(z)\}, \tag{10·6}$$

where

$$R_1(z) = \left(\frac{\partial A}{\partial z}\right)_{t=\tau} = \sum_{s=1}^{\infty} A_s'(z)\,\frac{\tau^{2s}}{(2s)!}, \quad R_2(z) = \left(\frac{\partial^2 A}{\partial z\,\partial t}\right)_{t=\tau} = \sum_{s=1}^{\infty} A_s'(z)\,\frac{\tau^{2s-1}}{(2s-1)!}. \tag{10·7}$$

Only if $R(z)$ vanished identically would $L_j(z)$ be a solution of (5·11), and this is not the case.

Let us denote by $W_j(z)$ the solution of (5·11) with the conditions

$$\left.\begin{aligned} W_j(a_j) = L_j(a_j), \quad W_j'(a_j) = L_j'(a_j), \quad &\text{if } a_j \text{ is finite,} \\ W_j(z) \sim L_j(z) \quad \text{as} \quad z \to a_j, \quad &\text{if } a_j \text{ is finite.} \end{aligned}\right\} \tag{10·8}$$

We shall prove theorem B by showing that for large u the difference $W_j(z) - L_j(z)$ is asymptotically negligible compared with the error term in (5·12). Only a proof for $j = 1$ need be recorded; the other two cases are exactly similar.

From (5·11) and (10·5) we derive

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\{W_1(z) - L_1(z)\} - \{uz + f(z)\}\{W_1(z) - L_1(z)\} = P_1'(vz)\,R(z). \tag{10·9}$$

Let us define a sequence of functions $\{h_n(z)\}$ by the relations $h_0(z) = 0$ and

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}h_n(z) - uzh_n(z) = f(z)\,h_{n-1}(z) + P_1'(vz)\,R(z) \quad (n \geqslant 1), \tag{10·10}$$

with the boundary values

$$h_n(a_1) = h_n'(a_1) = 0. \tag{10·11}$$

Suppose first that z lies in the part of $\mathbf{D}_1$ common to the sector $\mathbf{S}_1 + \mathbf{S}_2$ (figure 1). Applying the principle of variation of parameters to (10·10) and using (4·7), we find that

$$h_n(z) = \frac{2\pi\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}}{v}\int_z^{a_1}\{P_1(vz)\,P_2(vt) - P_2(vz)\,P_1(vt)\}\{f(t)\,h_{n-1}(t) + P_1'(vt)\,R(t)\}\,\mathrm{d}t, \tag{10·12}$$

if $n \geqslant 1$. Putting $n = 1$ and substituting (10·6), we obtain

$$h_1(z) = \mathrm{e}^{-\tau\sqrt{u}}\exp\{-\tfrac{2}{3}(vz)^{\frac{3}{2}}\}\int_z^{a_1}\psi(z, v, t)\,\{R_1(t) + u^{-\frac{1}{2}}R_2(t)\}\,\mathrm{d}t, \tag{10·13}$$

where

$$\psi(z, v, t) = 4\pi\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}\{P_1(vz)\,P_2(vt) - P_2(vz)\,P_1(vt)\}\,P_1'(vt)\exp\{\tfrac{2}{3}(vz)^{\frac{3}{2}}\}. \tag{10·14}$$

In consequence of the definition of the domain $\mathbf{D}_1$ given in § 5 we may select a path of integration in (10·12) and (10·13) lying wholly in $\mathbf{D}'$ and sector $\mathbf{S}_1 + \mathbf{S}_2$, such that if t is any point on the path

$$|\exp(-\tfrac{2}{3}t^{\frac{3}{2}})| \leqslant |\exp(-\tfrac{2}{3}z^{\frac{3}{2}})|. \tag{10·15}$$

Using (4·5), (4·6) and (10·14), we deduce that

$$|\psi(z,v,t)| < \frac{A}{1+|vz|^{\frac{1}{4}}} + \frac{A\,|\exp\{\frac{4}{3}(vz)^{\frac{3}{2}} - \frac{4}{3}(vt)^{\frac{3}{2}}\}|}{1+|vz|^{\frac{1}{4}}},$$

where the symbol A is used generically to denote a constant independent of z, u and t. If (10·15) is substituted in the last result, it is seen that

$$|\psi(z,v,t)| < A(1+|vz|^{\frac{1}{4}})^{-1}, \tag{10·16}$$

and substituting this result in (10·13), we obtain

$$|h_1(z)| < A\,\mathrm{e}^{-\tau\sqrt{u}}\,\frac{|\exp\{-\frac{2}{3}(vz)\}^{\frac{3}{2}}|}{1+|vz|^{\frac{1}{4}}}\int_z^{a_1}|R_1(t)+u^{-\frac{1}{2}}R_2(t)|\,|\mathrm{d}t|. \tag{10·17}$$

From (10·7) and lemma 3 it follows that

$$R_1(z) = O(|z|^{-\frac{3}{2}}), \quad R_2(z) = O(|z|^{-\frac{3}{2}}), \tag{10·18}$$

as $|z|\to\infty$ in $\mathbf{D}'$, uniformly with respect to $\arg z$. The path of integration in (10·12), (10·13) and (10·17) may be confined to the following contours and still satisfy the conditions above:

(i) the level curves of $\exp(-\frac{2}{3}z^{\frac{3}{2}})$ through z and a_1,
(ii) the rays $\arg z = 0, -\frac{2}{3}\pi$,
(iii) the boundary of $\mathbf{D}'$.

With the aid of (4·9) and (10·18), it may be shown that on such paths

$$\int_z^{a_1}|R_1(t)\,\mathrm{d}t| < r_1, \quad \int_z^{a_1}|R_2(t)\,\mathrm{d}t| < r_2, \tag{10·19}$$

where r_1, r_2 are assignable constants independent of z and, of course, u. Substituting these inequalities in (10·17) and absorbing the factor $r_1+u^{-\frac{1}{2}}r_2$ in the generic constant A, we obtain

$$|h_1(z)| < A\,\mathrm{e}^{-\tau\sqrt{u}}\,(1+|vz|^{\frac{1}{4}})^{-1}\,|\exp\{-\tfrac{2}{3}(vz)^{\frac{3}{2}}\}|. \tag{10·20}$$

Next, from (10·12), we obtain

$$h_{n+1}(z)-h_n(z) = \frac{1}{v}\exp\{-\tfrac{2}{3}(vz)^{\frac{3}{2}}\}\int_z^{a_1}\chi(z,v,t)f(t)\,\frac{\{h_n(t)-h_{n-1}(t)\}}{\exp\{-\frac{2}{3}(vt)^{\frac{3}{2}}\}}\,\mathrm{d}t, \tag{10·21}$$

provided $n\geq 1$, where

$$\chi(z,v,t) = 2\pi\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}\{P_1(vz)\,P_2(vt)-P_2(vz)\,P_1(vt)\}\exp\{\tfrac{2}{3}(vz)^{\frac{3}{2}}-\tfrac{2}{3}(vt)^{\frac{3}{2}}\}.$$

Using (4·5) and (10·15) we readily show (cf. (10·16)) that

$$|\chi(z,v,t)| < A(1+|vz|^{\frac{1}{4}})^{-1}. \tag{10·22}$$

Let us assume temporarily that

$$|h_n(z)-h_{n-1}(z)| < k_n(v)\,\mathrm{e}^{-\tau\sqrt{u}}\,(1+|vz|^{\frac{1}{4}})^{-1}\,|\exp\{-\tfrac{2}{3}(vz)^{\frac{3}{2}}\}|,$$

where $k_n(v)$ is independent of z. Then from (10·21) and (10·22) we obtain

$$|h_{n+1}(z)-h_n(z)| < \frac{A}{v}\,k_n(v)\,\mathrm{e}^{-\tau\sqrt{u}}\,\frac{|\exp\{-\frac{2}{3}(vz)^{\frac{3}{2}}\}|}{1+|vz|^{\frac{1}{4}}}\int_z^{a_1}|f(t)\,\mathrm{d}t|.$$

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Now from (5·1) we have $f(z) = O(|z|^{-2})$ as $|z|\to\infty$, and so $\int_z^{a_1} |f(t)\,dt|$ taken along the admissible paths of integration is a bounded function of z (cf. (10·19)), and may be absorbed in the constant A. Hence, by induction using (10·20), we deduce that

$$|h_{n+1}(z)-h_n(z)| < A\left(\frac{A}{v}\right)^n e^{-\tau\sqrt{u}} \frac{|\exp\{-\frac{2}{3}(vz)^{\frac{3}{2}}\}|}{1+|vz|^{\frac{1}{4}}}. \tag{10·23}$$

Thus when $v > A$, the series

$$\sum_{n=0}^{\infty} \{h_{n+1}(z)-h_n(z)\}$$

converges uniformly with respect to z. From (10·8), (10·9), (10·10) and (10·11) it follows that its sum is $W_1(z)-L_1(z)$. Moreover, if $v \geqslant 2A$, we have

$$|W_1(z)-L_1(z)| = \left|\sum_{n=0}^{\infty}\{h_{n+1}(z)-h_n(z)\}\right| < A\,e^{-\tau\sqrt{u}} \frac{|\exp\{-\frac{2}{3}(vz)^{\frac{3}{2}}\}|}{1+|vz|^{\frac{1}{4}}} \sum_{n=0}^{\infty}\frac{1}{2^n}$$

$$= 2A\,e^{-\tau\sqrt{u}} \frac{|\exp\{-\frac{2}{3}(vz)^{\frac{3}{2}}\}|}{1+|vz|^{\frac{1}{4}}}, \tag{10·24}$$

and so for large u, $W_1(z)-L_1(z)$ is asymptotically negligible compared with the error term in (5·12). This completes the proof of (5·12) for $j = 1$ when z lies in the part of $\mathbf{D}_1$ common to the sector $\mathbf{S}_1+\mathbf{S}_2$. A similar proof holds when z lies in the part of $\mathbf{D}_1$ common to $\mathbf{S}_1+\mathbf{S}_3$, the complementary solutions of (10·10) being taken to be $P_1(vz)$ and $P_3(vz)$, in place of $P_1(vz)$ and $P_2(vz)$.

For the derivative we derive, from (10·12) and (10·6),

$$h_1'(z) = v\,e^{-\tau\sqrt{u}} \exp\{-\tfrac{2}{3}(vz)^{\frac{3}{2}}\} \int_z^{a_1} \psi_1(z,v,t)\,\{R_1(t)+u^{-\frac{1}{2}}R_2(t)\}\,dt, \tag{10·25}$$

and

$$h_{n+1}'(z)-h_n'(z) = \exp\{-\tfrac{2}{3}(vz)^{\frac{3}{2}}\} \int_z^{a_1} \chi_1(z,v,t)\,f(t)\,\frac{\{h_n(t)-h_{n-1}(t)\}}{\exp\{-\frac{2}{3}(vt)^{\frac{3}{2}}\}}\,dt, \tag{10·26}$$

if $n \geqslant 1$, where

$$\psi_1(z,v,t) = 4\pi\,e^{\frac{1}{3}\pi i}\{P_1'(vz)\,P_2(vt)-P_2'(vz)\,P_1(vt)\}\,P_1'(vt)\exp\{\tfrac{2}{3}(vz)^{\frac{3}{2}}\},$$

$$\chi_1(z,v,t) = 2\pi\,e^{\frac{1}{3}\pi i}\{P_1'(vz)\,P_2(vt)-P_2'(vz)\,P_1(vt)\}\exp\{\tfrac{2}{3}(vz)^{\frac{3}{2}}-\tfrac{2}{3}(vt)^{\frac{3}{2}}\}.$$

We readily show (cf. (10·16) and (10·22)) that

$$|\psi_1(z,v,t)|,\ |\chi_1(z,v,t)| < A(1+|vz|^{\frac{1}{2}}), \tag{10·27}$$

and thence, using also (10·19) and (10·23), that

$$|h_1'(z)| < Av\,e^{-\tau\sqrt{u}}\,(1+|vz|^{\frac{1}{4}})\,|\exp\{-\tfrac{2}{3}(vz)^{\frac{3}{2}}\}|,$$

$$|h_{n+1}'(z)-h_n'(z)| < A(A/v)^n\,v\,e^{-\tau\sqrt{u}}\,(1+|vz|^{\frac{1}{4}})\,|\exp\{-\tfrac{2}{3}(vz)^{\frac{3}{2}}\}|.$$

Thus when $v \geqslant 2A$

$$|W_1'(z)-L_1'(z)| = \left|\sum_{n=0}^{\infty}\{h_{n+1}'(z)-h_n'(z)\}\right| < 2Av\,e^{-\tau\sqrt{u}}\,(1+|vz|^{\frac{1}{4}})\,|\exp\{-\tfrac{2}{3}(vz)^{\frac{3}{2}}\}|. \tag{10·28}$$

This result in conjunction with (10·4) establishes (5·13) when z lies in $\mathbf{D}_1$ and $\mathbf{S}_1+\mathbf{S}_2$. A similar proof holds when z lies in $\mathbf{D}_1$ and $\mathbf{S}_1+\mathbf{S}_3$.

This completes the proof of theorem B when the arbitrary constants in the second of (5·8) are prescribed by (9·3).

11. Existence of solutions when $A_1(c)$, $A_2(c)$, ..., do not vanish

Let $\phi(u)$ be an arbitrary function of u with the property

$$\phi(u) \sim 1+\sum_{s=1}^{\infty}\frac{\phi_s}{u^s},$$

as $u\to\infty$, where $\phi_1, \phi_2, \ldots$, are constants. Then $\phi(u)\,W_j(z)$ is a solution of (5·11), and from (5·14) and (5·15) we see that the asymptotic expansions of this function and its derivative for large u are given by

$$\phi(u)\,W_j(z) \sim P_j(vz)\sum_{s=0}^{\infty}\frac{A_s^*(z)}{u^s}+\frac{P_j'(vz)}{v^2}\sum_{s=0}^{\infty}\frac{B_s^*(z)}{u^s},$$

$$\phi(u)\,W_j'(z) \sim P_j(vz)\sum_{s=0}^{\infty}\frac{C_s^*(z)}{u^s}+vP_j'(vz)\sum_{s=0}^{\infty}\frac{D_s^*(z)}{u^s},$$

where

$$A_s^* = A_s+\phi_1 A_{s-1}+\ldots+\phi_{s-1}A_1+\phi_s, \qquad B_s^* = B_s+\phi_1 B_{s-1}+\ldots+\phi_{s-1}B_1+\phi_s B_0,$$
$$C_s^* = C_s+\phi_1 C_{s-1}+\ldots+\phi_{s-1}C_1+\phi_s C_0, \quad D_s^* = D_s+\phi_1 D_{s-1}+\ldots+\phi_{s-1}D_1+\phi_s.$$

With the aid of (3·6) and (3·9) with $g(z)=z$, we may immediately verify that A_s^*, B_s^*, C_s^* and D_s^* satisfy the same recurrence relations as A_s, B_s, C_s and D_s. Moreover, using (9·3), we obtain

$$A_s^*(c) = \phi_s.$$

In other words, dropping the stars, theorem B remains true if the arbitrary constants in the second of (5·8) have any prescribed values, provided there exists a function $\phi(u)$ with the property

$$\phi(u) \sim 1+\sum_{s=1}^{\infty}\frac{A_s(c)}{u^s}$$

for some point c of $\mathbf{D}'$. This is the result stated in § 5.

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